Stochastic AC Optimal Power Flow with Affine Recourse

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Dispatch model consists of two stages: **day-ahead** and **real-time**

- **Set of all uncertain parameter realizations**
- **Day-ahead dispatch (generator set-points)**
Dispatch model consists of two-stages: **day-ahead** and **real-time**

- **Set of all uncertain parameter realizations**
- **Day-ahead dispatch (generator set-points)**
- **Uncertain parameters are realized**
Dispatch model consists of two-stages: **day-ahead** and **real-time**

- **Set of all uncertain parameter realizations**
- **Day-ahead dispatch (generator set-points)**
- **Uncertain parameters are realized**
- **Real-time dispatch (a function of uncertain parameter realization)**
Two-Stage Economic Dispatch

Dispatch model consists of two-stages: **day-ahead** and **real-time**

For the purpose of this talk...

- Focus on the single-period (e.g., 1–2 PM)
Certainty equivalent approach:

(Stage 1) **Day-ahead**: schedule of gen. to meet *forecasted demand*
Power System Operations Today

Certainty equivalent approach:

(Stage 1)  **Day-ahead**: schedule of gen. to meet *forecasted demand*

Neglects the cost of recourse to correct imbalances in real-time
Real-time adjustments to compensate any residual imbalance between day-ahead scheduled supply and realized demand.

Certainty equivalent approach:

Stage 2: **Real-time** adjustments to compensate any residual imbalance between day-ahead scheduled supply and realized demand.
Certainty equivalent approach:

(Stage 2) **Real-time**: adjustment to **compensate any residual imbalance** between day-ahead scheduled supply and realized demand

Yields **minor inefficiencies in today’s operations** where load can be predicted fairly accurately
**Power System Operations Tomorrow**

Net-load cannot be predicted with high accuracy

\[ \text{Net-load} = \text{Load} - \text{Renewable supply} \]

- Net-load cannot be predicted with high accuracy
Power System Operations Tomorrow

Net-load = Load − Renewable supply

- Net-load cannot be predicted with high accuracy
- Certainty equivalent approach will result in a significant increase in the system operational cost, under high renewable generation
- Need for new optimization methods that account for the cost of recourse
Talk Outline

1. Stochastic AC-Optimal Power Flow (S-OPF)

2. (Conservative) Convex Inner Approximation

3. Feasibility Guarantees

4. 9-Bus Case Study
AC Power Flow Model

Consider an \( n \)-bus transmission network

- \( v_i \in \mathbb{C} \), complex bus voltage
- \( s_i \in \mathbb{C} \), complex net power injection
- \( s_{ij} \in \mathbb{C} \), complex power flow from bus-\( i \) to bus-\( j \),

for each bus \( i = 1, \ldots, n \)

Related according to **quadratic equations**

\[
\begin{align*}
  s_i &= v^* \Psi_i v & s_{ij} &= v^* \Phi_{ij} v
\end{align*}
\]

where \( \Psi_i \in \mathbb{C}^{n \times n} \) and \( \Phi_{ij} \in \mathbb{C}^{n \times n} \) are build from network admittance data

**Constraints:**

\[
\begin{align*}
  v_i &\leq |v_i| \leq \bar{v}_i & -l_{ij} &\leq \text{Re}\{s_{ij}\} \leq l_{ij}
\end{align*}
\]
Uncertainty Model

We model uncertainty according to the \textit{exogenous random vector} \[ \xi \in \mathbb{R}^k \]

- Captures uncertainty in power generation, e.g., from wind and solar
Uncertainty Model

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\[ \xi \in \mathbb{R}^k \]

- Captures uncertainty in power generation, e.g., from wind and solar
- 1\textsuperscript{st} and 2\textsuperscript{nd} moments are assumed known
- \( \xi \) is supported on a convex compact set \( \Xi \subseteq \mathbb{R}^k \)
Uncertainty Model

We model uncertainty according to the \textbf{exogenous random vector}

\[ \xi \in \mathbb{R}^k \]

- Captures uncertainty in power generation, e.g., from wind and solar
- 1\textsuperscript{st} and 2\textsuperscript{nd} moments are assumed known
- \( \xi \) is supported on a convex compact set \( \Xi \subseteq \mathbb{R}^k \)

Set of possible realizations of \( \Xi \in \mathbb{R}^k \) defined as:

\[
\Xi = \bigcap_{i=1}^{p} \{ \xi \in \mathbb{R}^k \mid \xi^T A_i \xi + b_i^T \xi + c_i \leq 0 \}
\]

intersection of \( p \) compact convex sets \( (A_i \geq 0 \ \forall \ i) \)
Recall the **two-stage single-period** economic dispatch model.
Generator Model

For each generator $i = 1, \ldots, n$ (one at each bus) determine:

- **Day-ahead dispatch**: $g_i^{DA} \in \mathbb{C}$ (determined 24-36 hours in advance)
For each generator $i = 1, \ldots, n$ (one at each bus) determine:

- **Day-ahead dispatch**: $g_{i}^{\text{DA}} \in C$ (determined 24-36 hours in advance)
- **Real-time dispatch**: $g_{i}^{\text{RT}} : \Xi \rightarrow C$ (a function of $\xi$)
Generator Model

For each generator $i = 1, \ldots, n$ (one at each bus) determine:

- **Day-ahead dispatch**: $g_i^{DA} \in \mathbb{C}$ (determined 24-36 hours in advance)
- **Real-time dispatch**: $g_i^{RT} : \Xi \rightarrow \mathbb{C}$ (a function of $\xi$)

$$0 \leq g_i^{DA} \leq \gamma_i$$  \hspace{1cm} Nominal capacity constraint
For each generator $i = 1, \ldots, n$ (one at each bus) determine:

- **Day-ahead dispatch**: $g^\text{DA}_i \in \mathcal{C}$ (determined 24-36 hours in advance)
- **Real-time dispatch**: $g^\text{RT}_i : \Xi \rightarrow \mathcal{C}$ (a function of $\xi$)

\[
0 \leq g^\text{DA}_i \leq \gamma_i \quad \text{Nominal capacity constraint}
\]

\[
|g^\text{RT}_i(\xi) - g^\text{DA}_i| \leq \rho_i \quad \text{Ramping constraint}
\]
Generator Model

For each generator \( i = 1, \ldots, n \) (one at each bus) determine:

- **Day-ahead dispatch:** \( g_{i}^{\text{DA}} \in \mathcal{C} \) (determined 24-36 hours in advance)
- **Real-time dispatch:** \( g_{i}^{\text{RT}} : \Xi \rightarrow \mathcal{C} \) (a function of \( \xi \))

**Nominal capacity constraint**
\[
0 \leq g_{i}^{\text{DA}} \leq \gamma_{i}
\]

**Ramping constraint**
\[
\left| g_{i}^{\text{RT}}(\xi) - g_{i}^{\text{DA}} \right| \leq \rho_{i}
\]

**Stochastic capacity constraint**
\[
0 \leq g_{i}^{\text{RT}}(\xi) \leq \bar{g}_{i}^{\ast} \xi
\]
Generator Model

For each generator \( i = 1, \ldots, n \) (one at each bus) determine:

- **Day-ahead dispatch:** \( g_i^{\text{DA}} \in \mathcal{C} \) (determined 24-36 hours in advance)

- **Real-time dispatch:** \( g_i^{\text{RT}} : \Xi \rightarrow \mathcal{C} \) (a function of \( \xi \))

\[
0 \leq g_i^{\text{DA}} \leq \gamma_i \quad \text{Nominal capacity constraint}
\]

\[
|g_i^{\text{RT}}(\xi) - g_i^{\text{DA}}| \leq \rho_i \quad \text{Ramping constraint}
\]

\[
\forall \xi \in \Xi \quad 0 \leq g_i^{\text{RT}}(\xi) \leq \bar{g}_i^* \xi \quad \text{Stochastic capacity constraint}
\]

Robust constraint satisfaction
Examples of Generators

1. **Base load** ($\rho_i = 0$) – e.g., nuclear, coal

   \[ g_{i}^{RT}(\xi) = g_{i}^{DA} \]

2. **Peaking load** ($\rho_i = \infty$) – e.g., gas, petroleum

   \[ 0 \leq g_{i}^{RT}(\xi) \leq \gamma_i \]

3. **Variable renewable** – e.g., wind, solar

   \[ 0 \leq g_{i}^{RT}(\xi) \leq \bar{g}_i^* \xi \]
Stochastic AC Optimal Power Flow (S-OPF)

minimize

\[
E \left[ \sum_{i=1}^{n} c_i \ g_i^{RT}(\xi) \right]
\]

Expected cost of active power generation \( (c_i \geq 0) \)
minimize \[ \mathbb{E} \left[ \sum_{i=1}^{n} c_i \, g_i^{RT} (\xi) \right] \]

subject to

\[ 0 \leq g_i^{DA} \leq \gamma_i \]
\[ |g_i^{RT} (\xi) - g_i^{DA}| \leq \rho_i \]
\[ 0 \leq g^{RT} (\xi) \leq \bar{g}^* \xi \]

Generator constraints
Stochastic AC Optimal Power Flow (S-OPF)

minimize \[ \mathbb{E} \left[ \sum_{i=1}^{n} c_i g_i^{\text{RT}}(\xi) \right] \]

subject to

\[ 0 \leq g_i^{\text{DA}} \leq \gamma_i \]

\[ |g_i^{\text{RT}}(\xi) - g_i^{\text{DA}}| \leq \rho_i \]

\[ 0 \leq g^{\text{RT}}(\xi) \leq \bar{g}^* \xi \]

Bus voltage mag. constraints

\[ v_i \leq |v_i(\xi)| \leq \bar{v}_i, \quad \forall \ i = 1, \ldots, n \]
Stochastic AC Optimal Power Flow (S-OPF)

minimize \[ \mathbb{E} \left[ \sum_{i=1}^{n} c_i g_i^{RT}(\xi) \right] \]

subject to

\[ 0 \leq g_i^{DA} \leq \gamma_i \]

\[ |g_i^{RT}(\xi) - g_i^{DA}| \leq \rho_i \]

\[ 0 \leq g^{RT}(\xi) \leq \bar{g}^* \xi \]

\[ v_i \leq |v_i(\xi)| \leq \bar{v}_i, \quad \forall \ i = 1, \ldots, n \]

\[ |v(\xi)^* \Phi_{ij} v(\xi)| \leq \ell_{ij}, \quad \forall \ i, j = 1, \ldots, n \]
Stochastic AC Optimal Power Flow (S-OPF)

minimize \[ \mathbb{E} \left[ \sum_{i=1}^{n} c_i g_i^{RT}(\xi) \right] \]

subject to

\[ 0 \leq g_i^{DA} \leq \gamma_i \]
\[ |g_i^{RT}(\xi) - g_i^{DA}| \leq \rho_i \]
\[ 0 \leq g_i^{RT}(\xi) \leq \bar{g}^* \xi \]

\[ v_i \leq |v_i(\xi)| \leq \bar{v}_i, \quad \forall \ i = 1, \ldots, n \]
\[ |v(\xi)^* \Phi_{ij} v(\xi)| \leq \ell_{ij}, \quad \forall \ i, j = 1, \ldots, n \]

\[ g_i^{RT}(\xi) - d_i = v(\xi)^* \Psi_i v(\xi), \quad \forall \ i = 1, \ldots, n \]
Stochastic AC Optimal Power Flow (S-OPF)

minimize \[ \mathbb{E} \left[ \sum_{i=1}^{n} c_i g_{i}^{RT}(\xi) \right] \]

subject to

\[ 0 \leq g_{i}^{DA} \leq \gamma_i \]
\[ |g_{i}^{RT}(\xi) - g_{i}^{DA}| \leq \rho_i \]
\[ 0 \leq g^{RT}(\xi) \leq \bar{g}^{*} \xi \]
\[ v_{i} \leq |v_{i}(\xi)| \leq \bar{v}_{i}, \quad \forall i = 1, \ldots, n \]
\[ |v(\xi) \Phi_{ij} v(\xi)| \leq \ell_{ij}, \quad \forall i, j = 1, \ldots, n \]
\[ g_{i}^{RT}(\xi) - d_{i} = v(\xi) \Psi_i v(\xi), \quad \forall i = 1, \ldots, n \]

Decision variables: \[ g^{DA} \in \mathbb{C}^n, \quad g^{RT} : \mathbb{R}^k \to \mathbb{C}^n, \text{ and } v : \mathbb{R}^k \to \mathbb{C}^n \]
Stochastic AC Optimal Power Flow (S-OPF)

minimize \( \mathbb{E} \left[ \sum_{i=1}^{n} c_i \, g_{RT}^i (\xi) \right] \)

subject to
\[
\begin{align*}
0 &\leq g_{DA} \leq \gamma_i \\
| g_{RT}^i (\xi) - g_{DA}^i | &\leq \rho_i \\
0 &\leq g_{RT} (\xi) \leq \bar{g}^* \xi \\
v_i &\leq |v_i (\xi)| \leq \bar{v}_i, \quad \forall \ i = 1, \ldots, n \\
|v(\xi)^* \Phi_{ij} v(\xi)| &\leq \ell_{ij}, \quad \forall \ i, j = 1, \ldots, n \\
g_{RT}^i (\xi) - d_i = v(\xi)^* \Psi_i v(\xi), \quad \forall \ i = 1, \ldots, n
\end{align*}
\]

Decision variables:
\[
\begin{align*}
g_{DA} &\in \mathbb{C}^n, \\
g_{RT} : \mathbb{R}^k &\rightarrow \mathbb{C}^n, \quad \text{and} \quad v : \mathbb{R}^k &\rightarrow \mathbb{C}^n
\end{align*}
\]

Robust constraint satisfaction
\( \forall \ \xi \in \Xi \)
Reformulate S-OPF as Stochastic QCQP

Letting \( x \triangleq \begin{bmatrix} \Re\{g^{\text{DA}}\} \\ \Im\{g^{\text{DA}}\} \end{bmatrix} \in \mathbb{R}^{2n} \) we can rewrite the stochastic AC-OPF as:

\[
\begin{align*}
\text{minimize} & \quad \mathbb{E}[v(\xi) * P_0 v(\xi)] \\
\text{subject to} & \quad Ax \leq b \\
& \quad v(\xi)^* P_i v(\xi) + q_i^T x \leq r_i^T \xi, \quad \forall \ \xi \in \Xi \text{ and } i = 1, \ldots, m
\end{align*}
\]

- a stochastic quadratically constrained quadratic program (QCQP)
Reformulate S-OPF as Stochastic QCQP

Letting $x \triangleq \begin{bmatrix} \text{Re}\{g_{DA}\} \\ \text{Im}\{g_{DA}\} \end{bmatrix} \in \mathbb{R}^{2n}$ we can rewrite the stochastic AC-OPF as:

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\text{minimize} & \quad \mathbb{E} \left[ v(\xi)^* P_0 v(\xi) \right] \\
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& \quad v(\xi)^* P_i v(\xi) + q_i^T x \leq r_i^T \xi, \quad \forall \ \xi \in \Xi \text{ and } i = 1, \ldots, m
\end{align*}$$

- a stochastic quadratically constrained quadratic program (QCQP)

**Challenges:**

- NP-hard, in general (nonconvex quadratic objective and constraints)
- $\infty$ - dimensional decision space
- $\infty$ - number of constraints
Previous Work

1. Focus on linearized models of AC-power flow model (DC-OPF)

2. Affine or piecewise-affine approximations of the $\infty$-dim. decision space

3. Robust or chance-constrained formulations to treatment of uncertainty

- **Summers, Warrington, Morari, Lygeros [2014]**
  “Stochastic OPF based on convex approximations of chance constraints”

- **Warrington, Goulart, Mariethoz, Morari [2013]**
  “Policy-based reserves for power systems”

- **Jabr [2013]**
  “Adjustable robust OPF with renewable energy sources”

- **Lorca and Sun [2015]**
  “Adaptive robust optimization with dynamic uncertainty sets for multi-period economic dispatch under significant wind”
We treat the AC power flow model.

- **Vrakopoulou, Katsampani, Margellos, Lygeros, Andersson [2013]**
  “Probabilistic security-constrained AC-OPF”

- **Bai, Qu, Qiao, [2015]**
  “Robust AC-OPF for power networks with wind power generation”

- **Phan, Ghosh [2014]**
  “Two-stage stochastic optimization for OPF under renewable gen. uncertainty”
We develop a procedure to **approximate the S-OPF from within by a semidefinite program (SDP)**.
We develop a procedure to approximate the S-OPF from within by a semidefinite program (SDP).

Enables efficient calculation of feasible solutions to S-OPF.
Talk Outline

1. Stochastic AC-Optimal Power Flow (S-OPF)

2. (Conservative) Convex Inner Approximation
   - Approximation of $\infty$-dimensional decision space
   - Approximation of $\infty$-number of constraints
   - Addressing Nonconvexity

3. Feasibility Guarantees

4. 9-Bus Case Study
Approximation of $\infty$-Dimensional Decision Space

S-OPF is an optimization problem over an $\infty$-dim. decision space

\[
\begin{align*}
\text{minimize} & \quad \mathbb{E}[v(\xi)^*P_0v(\xi)] \\
\text{subject to} & \quad Ax \leq b \\
& \quad v(\xi)^*P_i v(\xi) + q_i^*x \leq r_i^*\xi, \quad \forall \ \xi \in \Xi \ \text{and} \ i = 1, \ldots, m
\end{align*}
\]
S-OPF is an optimization problem over an $\infty$-dim. decision space

\[
\begin{align*}
\text{minimize} & \quad \mathbb{E} [v(\xi)^* P_0 v(\xi)] \\
\text{subject to} & \quad Ax \leq b \\
& \quad v(\xi)^* P_i v(\xi) + q_i^* x \leq r_i^* \xi, \quad \forall \, \xi \in \Xi \text{ and } i = 1, \ldots, m
\end{align*}
\]

Restricting the function $v$ to be affine, i.e.,

\[
v(\xi) = V\xi, \quad \text{where} \quad V \in \mathbb{C}^{n \times k}
\]

yields an inner approximation to S-OPF over a finite-dim. decision space
S-OPF is an optimization problem over an $\infty$-dim. decision space

\[
\begin{align*}
& \text{minimize} & & \mathbb{E} [\xi^* (V^* P_0 V) \xi] \\
& \text{subject to} & & Ax \leq b \\
& & & \xi^* (V^* P_i V) \xi + q_i^* x \leq r_i^* \xi, \quad \forall \ \xi \in \Xi \text{ and } i = 1, \ldots, m
\end{align*}
\]

Restricting the function $\nu$ to be affine, i.e.,

\[
\nu(\xi) = V \xi, \quad \text{where } V \in \mathbb{C}^{n \times k}
\]

yields an inner approximation to S-OPF over a finite-dim. decision space
Approximation of $\infty$-Dimensional Decision Space

S-OPF is an optimization problem over an $\infty$-dim. decision space

\[
\begin{align*}
\text{minimize} \quad & \mathbb{E} [\xi^* (V^* P_0 V) \xi] = \text{tr} \left( \mathbb{E} [\xi \xi^*] V^* P_0 V \right) \\
\text{subject to} \quad & Ax \leq b \\
& \xi^* (V^* P_i V) \xi + q_i^* x \leq r_i^* \xi, \quad \forall \xi \in \Xi \text{ and } i = 1, \ldots, m
\end{align*}
\]

Restricting the function $\nu$ to be affine, i.e.,

\[
\nu(\xi) = V \xi, \quad \text{where} \quad V \in \mathbb{C}^{n \times k}
\]

yields an inner approximation to S-OPF over a finite-dim. decision space
Approximation of \( \infty \)-Dimensional Decision Space

S-OPF is an optimization problem over an \( \infty \)-dim. decision space

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(MV^*P_0V) \\
\text{subject to} & \quad Ax \leq b \\
& \quad \xi^*(V^*P_iV)\xi + q^*_i x \leq r^*_i \xi, \quad \forall \ \xi \in \Xi \quad \text{and} \quad i = 1, \ldots, m
\end{align*}
\]

Restricting the function \( \nu \) to be affine, i.e.,

\[\nu(\xi) = V\xi, \quad \text{where} \quad V \in \mathbb{C}^{n \times k}\]

yields an inner approximation to S-OPF over a finite-dim. decision space

a semi-infinite nonconvex QCQP
1. Stochastic AC-Optimal Power Flow

2. (Conservative) Convex Inner Approximation
   - Approximation of $\infty$-dimensional decision space
   - Approximation of $\infty$-number of constraints
   - Addressing Nonconvexity

3. Feasibility Guarantees

4. 9-Bus Case Study
Recall that the uncertainty set is described by

$$\Xi = \bigcap_{j=1}^{p} \{ \xi \in \mathbb{R}^k \mid \xi^T A_j \xi + b_j^T \xi + c_j \leq 0 \}.$$
Recall that the uncertainty set is described by

\[ \Xi = \bigcap_{j=1}^{p} \{ \xi \in \mathbb{R}^k \mid \xi^T A_j \xi + b_j^T \xi + c_j \leq 0 \} . \]

**Lemma (S-procedure):** If there exist \((x, V, y) \in \mathbb{R}^{2n} \times C^{n \times k} \times \mathbb{R}^p\) such that

\[
\begin{bmatrix}
-V^* P_i V & \frac{1}{2} r_i^* \\
\frac{1}{2} r_i & -q_i^* x
\end{bmatrix}
+ \sum_{j=1}^{p} y_j
\begin{bmatrix}
A_j & \frac{1}{2} b_j^* \\
\frac{1}{2} b_j & c_j
\end{bmatrix}
\succeq 0
\]

\[ y_1, \ldots, y_p \geq 0 \]

then \( \xi^* (V^* P_i V) \xi + q_i^* x \leq r_i^* \xi, \ \forall \ \xi \in \Xi . \)
Recall that the uncertainty set is described by

$$\Xi = \bigcap_{j=1}^{p} \{ \xi \in \mathbb{R}^k \mid \xi^T A_j \xi + b_j^T \xi + c_j \leq 0 \}.$$  

**Lemma (S-procedure):** If there exist \((x, V, y) \in \mathbb{R}^{2n} \times \mathbb{C}^{n \times k} \times \mathbb{R}^p\) such that

$$\begin{bmatrix} -V^* P_i V & \frac{1}{2} r_i^* \cr \frac{1}{2} r_i^* & -q_i^* x \end{bmatrix} + \sum_{j=1}^{p} y_j \begin{bmatrix} A_j & \frac{1}{2} b_j^* \cr \frac{1}{2} b_j^* & c_j \end{bmatrix} \preceq 0$$

$$y_1, \ldots, y_p \geq 0$$

then \(\xi^* (V^* P_i V) \xi + q_i^* x \leq r_i^* \xi, \ \forall \ \xi \in \Xi\).

$$\max_{\xi \in \Xi} \{ \xi^* (V^* P_i V) \xi + q_i^* x - r_i^* \xi \} \leq 0$$
Approximation of $\infty$-Number of Constraints

Recall that the uncertainty set is described by

$$\Xi = \bigcap_{j=1}^{p} \{ \xi \in \mathbb{R}^k \mid \xi^T A_j \xi + b_j^T \xi + c_j \leq 0 \}.$$ 

**Lemma (S-procedure):** If there exist $(x, V, y) \in \mathbb{R}^{2n} \times \mathbb{C}^{n \times k} \times \mathbb{R}^p$ such that

$$\begin{bmatrix}
-V^* P_i V & \frac{1}{2} r_i \\
\frac{1}{2} r_i^* & -q_i^* x
\end{bmatrix} + \sum_{j=1}^{p} y_j \begin{bmatrix} A_j & \frac{1}{2} b_j \\ \frac{1}{2} b_j^* & c_j \end{bmatrix} \succeq 0$$

express compactly by

$$\mathcal{L}(x, V^* P_i V, y) \succeq 0$$

then $\xi^*(V^* P_i V) \xi + q_i^* x \leq r_i^* \xi$, $\forall \xi \in \Xi$. 

14
Thus, the semi-infinite nonconvex QCQP

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(MV^*P_0V) \\
(x, V) & \quad A x \leq b \\
\text{subject to} & \quad \xi^*(V^*P_i V)\xi + q_i^* x \leq r_i^* \xi, \quad \forall \ \xi \in \Xi \quad \text{and} \quad i = 1, \ldots, m
\end{align*}
\]

admits an \textbf{inner approximation} as a finite-dim. optimization problem

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(MV^*P_0V) \\
(x, V, y^1, \ldots, y^m) & \quad A x \leq b \\
\text{subject to} & \quad \mathcal{L}(x, V^*P_i V, y^i) \geq 0, \quad \forall \ i = 1, \ldots, m
\end{align*}
\]
Thus, the semi-infinite nonconvex QCQP

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(MV^*P_0V) \\
\text{subject to} & \quad Ax \leq b \\
\xi^*(V^*P_iV)\xi + q_i^*x & \leq r_i^*\xi, \quad \forall \; \xi \in \Xi \text{ and } i = 1, \ldots, m
\end{align*}
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admits an \textbf{inner approximation} as a finite-dim. optimization problem

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(MV^*P_0V) \\
\text{subject to} & \quad Ax \leq b \\
\mathcal{L}(x, V^*P_iV, y^i) & \geq 0, \quad \forall \; i = 1, \ldots, m
\end{align*}
\]

- a nonconvex quadratically constrained quadratic prog. \((P_i \not\equiv 0)\)
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2. (Conservative) Convex Inner Approximation
   - Approximation of $\infty$-dimensional decision space
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3. Feasibility Guarantees

4. 9-Bus Case Study
Basic idea: bound $v^* P v$ from above by a convex quadratic function
Basic idea: bound $v^*Pv$ from above by a convex quadratic function

1. Decompose $P$ as:

$$P = P^+ + P^-$$

$suc 0$ $\suc 0$
Difference of Convex Functions

Basic idea: bound $v^* P v$ from above by a convex quadratic function

1. Decompose $P$ as:

$$ P = P^+ + P^- $$

2. Linearize the concave quadratic $v^* P^- v$ around a point $v_0 \in \mathbb{C}^n$ yields

$$ v^* P v = v^* P^+ v + v^* P^- v $$

$$ \leq v^* P^+ v + 2v_0^* P^- v - v_0^* P^- v_0 $$

convex quadratic

affine
Similar analysis in matrix space: Let $V_0 \in \mathbb{C}^{n \times k}$ be a given matrix.

$$V^* P V \xrightarrow{\phi V_0} V^* P^+ V + V_0^* P^- V + V^* P^+ V_0 - V_0^* P^- V_0$$

- convex quadratic
- affine
Similar analysis in matrix space: Let \( V_0 \in \mathbb{C}^{n \times k} \) be a given matrix.

\[
\begin{align*}
V^*PV &\quad \phi V_0 \\
\xrightarrow{\quad} &\quad V^*P^+V + V_0^*P^-V + V^*P^0V - V_0^*P^-V_0 \\
\quad &\quad \text{convex quadratic}
\quad &\quad \text{affine}
\end{align*}
\]

The nonconvex problem approximating S-OPF from within

\[
\begin{align*}
\text{minimize} &\quad \text{tr}(MV^*P_0V) \\
\text{subject to} &\quad Ax \leq b \\
&\quad \mathcal{L}(x, V^*P_iV, y^i) \geq 0, \quad \forall i = 1, \ldots, m
\end{align*}
\]
Similar analysis in matrix space: Let $V_0 \in \mathbb{C}^{n \times k}$ be a given matrix.

\[
V^* P V \quad \phi_{V_0} \quad \rightarrow \quad V^* P^+ V + V_0^* P^- V + V^* P^- V_0 - V_0^* P^- V_0
\]

convex quadratic

affine

The nonconvex problem approximating S-OPF from within can be approximated from within by convex problem

\[
\text{minimize} \quad \text{tr}(M \phi_{V_0}(V^* P_0 V))
\]

subject to

\[
Ax \leq b
\]

\[
\mathcal{L}(x, \phi_{V_0}(V^* P_i V), y^i) \geq 0, \quad \forall \ i = 1, \ldots, m
\]
Similar analysis in matrix space: Let $V_0 \in \mathbb{C}^{n \times k}$ be a given matrix.

\[
V^* P V \quad \xrightarrow{\phi V_0} \quad V^* P^+ V + V_0^* P^- V + V^* P^- V_0 - V_0^* P^- V_0
\]

The nonconvex problem approximating S-OPF from within can be approximated from within by convex problem

\[
\begin{align*}
\text{minimize} & \quad \text{tr} (M \phi V_0 (V^* P_0 V)) \\
\text{subject to} & \quad Ax \leq b \\
& \quad \mathcal{L}(x, \phi V_0 (V^* P_i V), y^i) \geq 0, \quad \forall \ i = 1, \ldots, m
\end{align*}
\]

Amounts to an SDP
Theorem: Let $V_0 \in \mathbb{C}^{n \times k}$ be a given matrix. Consider the SDP:

$$\begin{align*}
\text{minimize} & \quad \text{tr}(M \phi_{V_0}(V^* P_0 V)) \\
\text{subject to} & \quad Ax \leq b \\
& \quad \mathcal{L}(x, \phi_{V_0}(V^* P_i V), y^i) \geq 0, \quad \forall i = 1, \ldots, m
\end{align*}$$
SDP Inner Approximation to S-OPF

**Theorem:** Let $V_0 \in \mathbb{C}^{m \times k}$ be a given matrix. Consider the SDP:

```
minimize \( \text{tr}(M \phi_{V_0}(V^* P_0 V)) \)
subject to \( Ax \leq b \)
\( \mathcal{L}(x, \phi_{V_0}(V^* P_i V), y^i) \geq 0, \quad \forall i = 1, \ldots, m \)
```

Given a feasible solution \((x, V, y^1, \ldots, y^m)\) to the SDP, it holds that

\((x, V\xi)\)

is a feasible solution to the stochastic AC-OPF problem.
**Theorem:** Let $V_0 \in \mathbb{C}^{m \times k}$ be a given matrix. Consider the SDP:

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(M\phi_{V_0}(V^*P_0V)) \\
\text{subject to} & \quad Ax \leq b \\
& \quad \mathcal{L}(x, \phi_{V_0}(V^*P_iV), y^i) \geq 0, \quad \forall i = 1, \ldots, m
\end{align*}
\]

Given a feasible solution $(x, V, y^1, \ldots, y^m)$ to the SDP, it holds that $(x, V \xi)$ is a feasible solution to the stochastic AC-OPF problem.

**Question:** Is the SDP nonempty for any $V_0$?
Outline

1. Stochastic AC-Optimal Power Flow

2. (Conservative) Convex Inner Approximation

3. Feasibility Guarantees

4. 9-Bus Case Study
Feasibility Guarantees

Define the **minimum guaranteed production capacity** at bus $i$ as

$$\theta_i \triangleq \min_{\xi \in \Xi} \bar{g}_i^* \xi$$
Feasibility Guarantees

Define the **minimum guaranteed production capacity** at bus $i$ as

$$\theta_i \triangleq \min_{\xi \in \Xi} \bar{g}_i^* \xi$$

Recall the stochastic AC-OPF problem,

\[
\begin{align*}
\text{minimize} & \quad \mathbb{E} \left[ \sum_{i=1}^{n} c_i \ g_i^{\text{RT}}(\xi) \right] \\
\text{subject to} & \quad \vdots \\
0 & \leq g_{i}^{\text{RT}}(\xi) \leq \bar{g}_i^* \xi, \quad \forall \ i = 1, \ldots, n \\
\vdots & \\
\end{align*}
\]

\[
\forall \ \xi \in \Xi
\]

**Decision variables:** $g^{\text{DA}} \in \mathbb{C}^n$, $g^{\text{RT}} : \mathbb{R}^k \rightarrow \mathbb{C}^n$, and $v : \mathbb{R}^k \rightarrow \mathbb{C}^n$
Feasibility Guarantees

Define the **minimum guaranteed production capacity** at bus $i$ as

$$\theta_i \triangleq \min_{\xi \in \Xi} \bar{g}_i^* \xi$$

Recall the stochastic AC-OPF problem,

$$\begin{aligned}
\text{minimize} & \quad \mathbb{E} \left[ \sum_{i=1}^{n} c_i \, g_{i}^{\text{RT}}(\xi) \right] \\
\text{subject to} & \quad \vdots \\
0 \leq g_{i}^{\text{RT}}(\xi) \leq \bar{g}_i^* \xi, & \quad \forall \ i = 1, \ldots, n \\
\vdots \\
\end{aligned}$$

**Decision variables:** $g^{\text{DA}} \in \mathbb{C}^n$, $g^{\text{RT}} : \mathbb{R}^k \to \mathbb{C}^n$, and $v : \mathbb{R}^k \to \mathbb{C}^n$
Feasibility Guarantees

Define the **minimum guaranteed production capacity** at bus $i$ as

$$\theta_i \triangleq \min_{\xi \in \Xi} \mathcal{g}^*_i \xi$$

Recall the stochastic AC-OPF problem,

\[
\begin{aligned}
\text{minimize} & \quad \mathbb{E} \left[ \sum_{i=1}^{n} c_i g^\text{RT}_i(\xi) \right] \\
\text{subject to} & \quad : \\
& \quad 0 \leq g^\text{RT}_i(\xi) \leq \mathcal{g}^*_i \xi, \quad \forall \ i = 1, \ldots, n \\
& \quad : \\
\end{aligned}
\]

Equivalent to a deterministic AC-OPF problem.

Decision variables: $g^\text{DA} \in \mathbb{C}^n$, $g^\text{RT} : \mathbb{R}^k \rightarrow \mathbb{C}^n$, and $v : \mathbb{R}^k \rightarrow \mathbb{C}^n$
**Proposition:** Let $(g, v)$ be a feasible solution to the deterministic AC-OPF problem

\[
\begin{align*}
\text{minimize} \quad & \sum_{i=1}^{n} c_i g_i \\
\text{subject to} \quad & 0 \leq g_i \leq \theta_i, \quad \forall \ i = 1, \ldots, n \\
& v_i \leq |v_i| \leq \overline{v}_i, \quad \forall \ i = 1, \ldots, n \\
& |v^* \Phi_{ij} v| \leq \ell_{ij}, \quad \forall \ i, j = 1, \ldots, n \\
& g_i - d_i = v^* \Psi_i v, \quad \forall \ i = 1, \ldots, n
\end{align*}
\]
Proposition: Let \( (g, v) \) be a feasible solution to the deterministic AC-OPF problem

\[
\begin{align*}
&\text{minimize} & & \sum_{i=1}^{n} c_i g_i \\
&\text{subject to} & & 0 \leq g_i \leq \theta_i & \forall \ i = 1, \ldots, n \\
& & & v_i \leq |v_i| \leq \bar{v}_i, & \forall \ i = 1, \ldots, n \\
& & & |v^* \Phi_{ij} v| \leq \ell_{ij}, & \forall \ i, j = 1, \ldots, n \\
& & & g_i - d_i = v^* \Psi_i v, & \forall \ i = 1, \ldots, n 
\end{align*}
\]

Then, linearizing the concave part of \( V^* P_i V, \ i = 0, \ldots, m \) around

\[
V_0 = \begin{bmatrix} v & 0_{n \times (k-1)} \end{bmatrix}
\]

yields a nonempty SDP inner approximation to stochastic AC-OPF.
Successive SDP Inner Approximations

Let $v$ be a solution to the deterministic AC-OPF we considered and

$$V_0 = \begin{bmatrix} v & 0_{n \times (k-1)} \end{bmatrix}$$

Consider the SDP inner approximation to Stochastic AC-OPF

$$\begin{array}{c}
\text{minimize} & \text{tr}(M \phi_{V_0}(V^* P_0 V)) \\
\text{subject to} & Ax \leq b \\
& \mathcal{L}(x, \phi_{V_0}(V^* P_i V), y^i) \geq 0, \quad \forall i = 1, \ldots, m
\end{array}$$

1. SDP has a non-empty feasible set
Successive SDP Inner Approximations

Let $v$ be a solution to the deterministic AC-OPF we considered and

$$V_0 = \begin{bmatrix} v & 0_{n \times (k-1)} \end{bmatrix}$$

Consider the SDP inner approximation to Stochastic AC-OPF

$$(x, V, y^1, \ldots, y^m)$$

minimize $\text{tr}(M \phi_{V_0}(V^*P_0V))$

subject to $Ax \leq b$

$$\mathcal{L}(x, \phi_{V_0}(V^*P_iV), y^i) \succeq 0, \quad \forall i = 1, \ldots, m$$

1. SDP has a non-empty feasible set

2. **Question:** Can we use the solution to this SDP to improve upon the inner approximation?
Consider the iterative algorithm, where $V_0 \in \mathbb{C}^{n \times k}$ is given

\begin{align*}
\text{for } j = 0, 1, \ldots, \text{ solve the SDP:} \\
(x_{j+1}, V_{j+1}, \{y_{j+1}\}_i) \in \arg\min \quad & \text{tr}(M\phi_{V_j}(V^*P_0V)) \\
\text{subject to} \quad & Ax \leq b \\
& \mathcal{L}(x, \phi_{V_j}(V^*P_iV), y^i) \geq 0, \ \forall \ i
\end{align*}

Properties:

1. Each iteration yields a **SDP with a non-empty feasible set**
Consider the iterative algorithm, where $V_0 \in \mathbb{C}^{n \times k}$ is given.

for $j = 0, 1, \ldots$, solve the SDP:

$$(x_{j+1}, V_{j+1}, \{y_{j+1}\}_{i=1}^m) \in \arg\min \quad \text{tr}(M \phi_{V_j}(V^*P_0V))$$

subject to

$$Ax \leq b$$

$$\mathcal{L}(x, \phi_{V_j}(V^*P_iV), y^i) \geq 0, \quad \forall \ i$$

end

Properties:

1. Each iteration yields a SDP with a non-empty feasible set.

2. Yields a sequence of feasible solutions with nonincreasing costs to the stochastic AC-OPF.
Algorithm Performance

Expected cost of dispatch vs. # of iterations

deterministic AC-OPF
Algorithm Performance

Expected cost of dispatch vs. # of iterations

- **Deterministic AC-OPF**
- **Affine dispatch policy**
1. Stochastic AC-Optimal Power Flow (S-OPF)

2. (Conservative) Convex Inner Approximation

3. Feasibility Guarantees

4. 9-Bus Case Study
Generators

- One Base load
  
  \[ c_b \ (\$/\text{MW}) \]

- Two Peaking
  
  \[ c_p \ (\$/\text{MW}) \]

- Six Variable Renewable
  
  0 (\$/\text{MW})
Let \( \xi_i \) denote the max. available active power capacity of renewable gen. \( i \)

\[ \xi_i = \mu_i + \sigma \delta_i, \quad \text{where } (\delta_1, \ldots, \delta_6) \in \Delta \]

- \( \mu_i \): mean active power capacity of renewable gen. \( i \)
- \( \sigma \): scale parameter
- \( \Delta \): base uncertainty set (ellipsoid)
Let $\xi_i$ denote the max. available active power capacity of renewable gen. $i$

$$\xi_i = \mu_i + \sigma \delta_i,$$

where $(\delta_1, \ldots, \delta_6) \in \Delta$

- $\mu_i$ : mean active power capacity of renewable gen. $i$
- $\sigma$ : scale parameter
- $\Delta$ : base uncertainty set (ellipsoid)
Let $\xi_i$ denote the max. available active power capacity of renewable gen. $i$

\[ \xi_i = \mu_i + \sigma \delta_i, \quad \text{where } (\delta_1, \ldots, \delta_6) \in \Delta \]

- $\mu_i$: mean active power capacity of renewable gen. $i$
- $\sigma$: scale parameter
- $\Delta$: base uncertainty set (ellipsoid)
Uncertainty Model

Let $\xi_i$ denote the max. available active power capacity of renewable gen. $i$

$$\xi_i = \mu_i + \sigma \delta_i, \quad \text{where } (\delta_1, \ldots, \delta_6) \in \Delta$$

- $\mu_i$: mean active power capacity of renewable gen. $i$
- $\sigma$: scale parameter
- $\Delta$: base uncertainty set (ellipsoid)
The Effect of Uncertainty

Expected cost of dispatch vs. $\sigma$ \[ c_b/c_p = 0.2 \]
The Effect of Uncertainty

Cost distribution vs. $\sigma$  

$\left[ \frac{c_b}{c_p} = 0.2 \right]$
The Effect of Uncertainty

Cost distribution vs. $\sigma$ 

$[c_b/c_p = 0.2]$
Summary and Future Research

1. Defined a two-stage stochastic AC-OPF problem
   • $\infty$ - dim. decision space
   • $\infty$ - number of constraints
   • Nonconvex
Summary and Future Research

1. Defined a two-stage stochastic AC-OPF problem
   - $\infty$ - dim. decision space
   - $\infty$ - number of constraints
   - Nonconvex

2. Developed a SDP inner approximation to stochastic AC-OPF
Summary and Future Research

1. Defined a two-stage stochastic AC-OPF problem
   • $\infty$ - dim. decision space
   • $\infty$ - number of constraints
   • Nonconvex

2. Developed a SDP inner approximation to stochastic AC-OPF

Future Research

1. Outer approximations to Stochastic AC-OPF
2. Multi-period stochastic AC-OPF
Questions?

Thank you!
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