A Hierarchy of Polyhedral Approximations of Robust Semidefinite Programs

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Robust Semidefinite Program

Consider the robust semidefinite program (SDP)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \sum_{i=1}^{k} \xi_i A_i(x) \in S^n_+, \quad \forall \xi \in \Xi,
\end{align*}
\]

where

- \( x \in \mathbb{R}^m \) is the decision variable
- \( A_i : \mathbb{R}^m \rightarrow S^n \) is an affine function of \( x \), and
- \( \Xi \subseteq \mathbb{R}^k \) is the uncertainty set, a convex compact set

Some notation:

- \( S^n (S^n_+) \) : space of \( n \times n \) symmetric (symmetric PSD) matrices
The uncertainty set $\Xi$ is defined as

$$\Xi \triangleq \{ \xi \in \mathbb{R}^k \mid \xi_1 = 1, \ B\xi \in K \},$$

where $K$ is a proper cone, e.g.,

- positive orthant
- second-order cone
- positive semidefinite cone

Taken from Jerome Malic's “Semidefinite Projections, regularization algorithms and polynomial optimization.”
The robust SDP is \textbf{NP-hard}, in general

\[
\begin{align*}
\text{minimize} & \quad c^\top x \\
\text{subject to} & \quad \sum_{i=1}^{k} \xi_i A_i(x) \in S^n_+, \quad \forall \xi \in \Xi,
\end{align*}
\]
The robust SDP is **NP-hard**, in general

\[
\begin{align*}
\text{minimize} & \quad c^\top x \\
\text{subject to} & \quad \sum_{i=1}^{k} \xi_i A_i(x) \in \mathbf{S}_+^n, \quad \forall \xi \in \Xi,
\end{align*}
\]

The constraint holds if and only if

\[
\min_{\xi \in \Xi} \lambda_{\min} \left( \sum_{i=1}^{k} \xi_i A_i(x) \right) \geq 0.
\]

\(\lambda_{\min} : \mathbf{S}^n \to \mathbf{R}\) is the minimum eigenvalue function
Approximate the robust SDP with a robust linear program (LP) by approximating the PSD cone by a polyhedral cone.

Robust LP:

\[
\begin{align*}
\text{minimize} & \quad c^\top x \\
\text{subject to} & \quad \sum_{i=1}^{k} \xi_i A_i(x) \in S_n^+, \quad \forall \xi \in \Xi,
\end{align*}
\]

by approximating the PSD cone by a polyhedral cone.
A Robust LP Approach

Approximate the robust SDP with a robust linear program (LP)

Robust LP:
minimize \( c^T x \)
subject to \( \sum_{i=1}^{k} \xi_i A_i(x) \subseteq S_{+}^{n}, \quad \forall \xi \in \Xi, \)

by approximating the PSD cone by a polyhedral cone

Robust LPs: Admit finite-dimensional reformulations as conic convex programs over the cone \( K \) characterizing uncertainty set \( \Xi \), e.g.

<table>
<thead>
<tr>
<th>( K )</th>
<th>Polyhedral Cone</th>
<th>Second-order Cone</th>
<th>Semidefinite Cone</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robust LP</td>
<td>LP</td>
<td>SOCP</td>
<td>SDP</td>
</tr>
</tbody>
</table>
Finite-Dim. Reformulations and Approximations

Exact Reformulations

• Ben-Tal, El-Ghaoui, Nemirovski, ['00]
  – Ξ is “Unstructured norm-bounded”

Inner Approximations

• Ben-Tal, El-Ghaoui, Nemirovski, ['00]
  – Ξ is “Structured norm-bounded”

• Scherer and Hol, ['06]
  – Ξ is described by polynomial matrix inequalities

Other related work

• Packard et al. ['93]
• El-Ghaoui et al. ['97]
• Scherer ['05]
• Dietz et al. ['08]
• Ben-Tal et al. ['02]
• Oishi et al. ['08]
Exact Reformulations

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Inner Approximations

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- Scherer and Hol, ['06]
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1. Introduction

2. Inner and Outer Polyhedral Hierarchies of the PSD Cone

3. Inner and Outer Hierarchies of Robust SDPs

4. Application: Robust Resistance Network Design Problem
Polyhedral Approximation of $S^n_+$

The positive semidefinite cone

$$S^n_+ = \bigcap_{u \neq 0} \{ X \in S^n \mid u^\top X u \geq 0 \}$$

is an infinite intersection of half-spaces in $S^n$. 

$S^n_+$
Outer Polyhedral Approximation of $S^n_+$

The positive semidefinite cone

$$S^n_+ = \bigcap_{u \neq 0} \{X \in S^n \mid u^T X u \geq 0\}$$

is an infinite intersection of half-spaces in $S^n$.

- A finite intersection of half-spaces yields an outer polyhedral cone to $S^n_+$.
The positive semidefinite cone

\[ \mathbb{S}_+^n = \bigcap_{u \neq 0} \{ X \in \mathbb{S}^n \mid u^T X u \geq 0 \} \]

is an infinite intersection of half-spaces in \( \mathbb{S}^n \)

- A finite intersection of half-spaces yields an outer polyhedral cone to \( \mathbb{S}_+^n \)

- The dual of the outer polyhedral cone is an inner polyhedral cone to \( \mathbb{S}_+^n \)

\[ \mathbb{S}_+^n \subseteq \text{polyhedral} \iff (\text{polyhedral})^* \subseteq (\mathbb{S}_+^n)^* = \mathbb{S}_+^n \]
Let $\Delta$ denote the boundary of the $\ell_1$ norm ball in $\mathbb{R}^n$

$$\Delta := \{ x \in \mathbb{R}^n \mid \|x\|_1 = 1 \}.$$ 

The PSD cone can be expressed as:

$$S^n_+ = \bigcap_{u \in \Delta} \{ X \in S^n \mid u^\top Xu \geq 0 \}$$
Construction of Outer Polyhedral Approximations

Let \( \Delta \) denote the boundary of the \( \ell_1 \) norm ball in \( \mathbb{R}^n \).

\[
\Delta := \{ x \in \mathbb{R}^n \mid \|x\|_1 = 1 \}.
\]

The PSD cone can be expressed as:

\[
\mathbb{S}_+^n = \bigcap_{u \in \Delta} \{ X \in \mathbb{S}^n \mid u^\top X u \geq 0 \}
\]

- An **outer polyhedral cone** to \( \mathbb{S}_+^n \) arises by a **discretization of \( \Delta \)**, i.e.,

\[
\mathbb{S}_+^n \subseteq \{ X \in \mathbb{S}^n \mid u^\top X u \geq 0, \text{ for some } u \in \Delta \}.
\]
A Discretization Scheme of $\Delta$

Fix $r \in \mathbb{N}$. Consider the following discretization of $\Delta \subseteq \mathbb{R}^n$:

$$\Delta_r := \{ u \in \Delta \mid 2^r u \in \mathbb{Z}^n \}$$
A Discretization Scheme of $\Delta$

Fix $r \in \mathbb{N}$. Consider the following discretization of $\Delta \subseteq \mathbb{R}^n$:

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Examples: $n = 2$

$$\Delta_0 = \left\{ \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} \right\}$$
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**Examples: $n = 2$**

$$\Delta_0 = \left\{ \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} \right\}$$

$$\Delta_1 = \Delta_0 \cup \left\{ \begin{bmatrix} \pm \frac{1}{2} \\ \pm \frac{1}{2} \end{bmatrix} \right\}$$
A Discretization Scheme of $\Delta$

Fix $r \in \mathbb{N}$. Consider the following discretization of $\Delta \subseteq \mathbb{R}^n$:

$$\Delta_r := \{ u \in \Delta \mid 2^r u \in \mathbb{Z}^n \}$$

**Examples:** $n = 2$

$$\Delta_0 = \left\{ \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} \right\} \quad \Delta_1 = \Delta_0 \cup \left\{ \begin{bmatrix} \pm \frac{1}{2} \\ \pm \frac{1}{2} \end{bmatrix} \right\} \quad \Delta_2 = \Delta_1 \cup \left\{ \begin{bmatrix} \pm \frac{1}{4} \\ \pm \frac{3}{4} \end{bmatrix}, \begin{bmatrix} \pm \frac{3}{4} \\ \pm \frac{1}{4} \end{bmatrix} \right\}$$
Fix $r \in \mathbb{N}$. Consider the following discretization of $\Delta \subseteq \mathbb{R}^n$:

$$\Delta_r := \{ u \in \Delta \mid 2^r u \in \mathbb{Z}^n \}$$

**Remarks:** For any $r \in \mathbb{N}$, it holds that

$$\Delta_r \subseteq \Delta_{r+1}$$
A Discretization Scheme of $\Delta$

Fix $r \in \mathbb{N}$. Consider the following discretization of $\Delta \subseteq \mathbb{R}^n$:

$$\Delta_r := \{ u \in \Delta \mid 2^r u \in \mathbb{Z}^n \}$$

**Remarks:** For any $r \in \mathbb{N}$, it holds that

$$\Delta_r \subseteq \Delta_{r+1}$$

**Some notation:** The set $\Delta_r$ has $p_r$ elements denoted by:

$$u_1, u_2, \ldots, u_{p_r}$$
Outer Polyhedral Hierarchies of $S^n_+$

A hierarchy of outer polyhedral cones to $S^n_+$ arises by the following family of polyhedral cones

\[ O^n_r := \bigcap_{u \in \Delta_r} \{ X \in S^n | u^\top Xu \geq 0 \} \]

where $r \in \mathbb{N}$. In particular

\[ O^n_0 \supseteq S^n_+ \]
A hierarchy of outer polyhedral cones to $S_+^n$ arises by the following family of polyhedral cones

$$O_r^n := \bigcap_{u \in \Delta_r} \{ X \in S^n | u^\top X u \geq 0 \}$$

where $r \in \mathbb{N}$. In particular

$$O_0^n \supseteq O_1^n \supseteq S_+^n$$

since $\Delta_0 \subseteq \Delta_1$. 
A *hierarchy of outer polyhedral cones* to $S^n_+$ arises by the following family of polyhedral cones

$$O_r^n := \bigcap_{u \in \Delta_r} \{ X \in S^n \mid u^\top X u \geq 0 \}$$

where $r \in \mathbb{N}$. In particular:

$$O_0^n \supseteq O_1^n \supseteq O_2^n \supseteq S^n_+$$

since $\Delta_1 \subseteq \Delta_2$.
A hierarchy of outer polyhedral cones to $S^n_+$ arises by the following family of polyhedral cones

$$O^n_r := \bigcap_{u \in \Delta_r} \{ X \in S^n \mid u^\top X u \geq 0 \}$$

where $r \in \mathbb{N}$. In particular:

$$O^n_0 \supseteq O^n_1 \supseteq O^n_2 \supseteq \cdots \supseteq S^n_+$$

since $\Delta_r \subseteq \Delta_{r+1}$, for all $r \in \mathbb{N}$. 
The dual cones to $O^n_r$ give a **hierarch of** $S^n_+$ **inner polyhedral cones** to $S^n_+$

\[ I^n_r = (O^n_r)^* = \text{cone} \{ u_1 u_1^\top, \ldots, u_p u_p^\top \} \]

where $r \in \mathbb{N}$. In particular:

\[ I^n_0 \subseteq S^n_+ \]
Inner Polyhedral Hierarchies of $S^n_+$

The dual cones to $O^n_r$ give a **hierarch of** $S^n_+$ **inner polyhedral cones** to $S^n_+$

$$I^n_r = (O^n_r)^* = \text{cone}\{u_1u_1^\top, \ldots, u_{p_r}u_{p_r}^\top\}$$

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$$I^n_0 \subseteq I^n_1 \subseteq S^n_+$$
The dual cones to $O^n_\mathbb{R}$ give a hierarch of inner polyhedral cones to $S^n_+$.

$$I^n_r = (O^n_\mathbb{R})^* = \text{cone}\{u_1u_1^\top, \ldots, u_pu_p^\top\}$$

where $r \in \mathbb{N}$. In particular:

$$I^n_0 \subseteq I^n_1 \subseteq I^n_2 \subseteq \cdots \subseteq S^n_+$$
Inner Polyhedral Hierarchies of $S^n_+$

The dual cones to $O^n_r$ give a hierarchy of $S^n_+$ inner polyhedral cones to $S^n_+$

$$I^n_r = (O^n_r)^* = \text{cone}\{uu_1^\top, \ldots, uu_p^\top\}$$

where $r \in \mathbb{N}$. In particular:

$$I^n_0 \subseteq I^n_1 \subseteq I^n_2 \subseteq \cdots \subseteq S^n_+$$

Examples

- $I^n_0$ : cone of nonnegative diagonal matrices

- $I^n_1$ : cone of diagonally dominant matrices with nonnegative diagonal entries.

Ahmadi et al.'16 : Application of $I^n_1$ to Sums of Squares Optimization.
Theorem: For each level $r \in \mathbb{N}$,

1. $O^n_r \supseteq O^n_{r+1} \supseteq S^n_+$ and
   $$\bigcap_{i \in \mathbb{N}} O^n_i = S^n_+$$

2. $I^n_r \subseteq I^n_{r+1} \subseteq S^n_+$ and
   $$\operatorname{cl} \left( \bigcup_{i \in \mathbb{N}} I^n_i \right) = S^n_+$$
**Theorem:** For each level $r \in \mathbb{N}$,

1. $O^n_r \supseteq O^n_{r+1} \supseteq S^n_+$ and
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   \]

[Braun, Fiorini, Pokutta, Steurer ’12] “Approximation limits of linear programs (beyond hierarchies).”

“It’s not possible to approximate SDPs arbitrarily well using small LPs”
Talk Outline

1. Introduction

2. Inner and Outer Polyhedral Hierarchies of the PSD Cone

3. Inner and Outer Hierarchies of Robust SDPs

4. Application: Robust Resistance Network Design Problem
Recall the outer and inner polyhedral cones approximating $S^n_+$

\[ I^n_0 \subseteq I^n_1 \subseteq \cdots \subseteq S^n_+ \subseteq \cdots \subseteq O^n_1 \subseteq O^n_0. \]
Outer Approximations to Robust SDP

Recall the outer and inner polyhedral cones approximating $\mathbb{S}_+^n$

$$
\mathbb{I}_0^n \subseteq \mathbb{I}_1^n \subseteq \cdots \subseteq \mathbb{S}_+^n \subseteq \cdots \subseteq \mathbb{O}_1^n \subseteq \mathbb{O}_0^n.
$$

For any $r \in \mathbb{N}$, the robust LP:

$$
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \sum_{i=1}^{k} \xi_i A_i(x) \in \mathbb{O}_r^n \quad \forall \xi \in \Xi,
\end{align*}
$$

is an outer approx. to the robust SDP.
Inner Approximations to Robust SDP

Recall the outer and inner polyhedral cones approximating $\mathbb{S}^+_n$

$$\mathbb{I}_0^n \subseteq \mathbb{I}_1^n \subseteq \cdots \subseteq \mathbb{S}^+_n \subseteq \cdots \subseteq \mathbb{O}^n_1 \subseteq \mathbb{O}^n_0.$$ 

For any $r \in \mathbb{N}$, the robust LP:

- minimize $c^\top x$
  - subject to $\sum_{i=1}^k \xi_i A_i(x) \in \mathbb{S}^+_n \quad \forall \xi \in \Xi$,

is an **outer approx.** to the robust SDP

- minimize $c^\top x$
  - subject to $\sum_{i=1}^k \xi_i A_i(x) \in \mathbb{S}^+_n \quad \forall \xi \in \Xi$,

is an **inner approx.** to the robust SDP
The **hyperplane representation** of the outer polyhedral cones $O^n_r$ gives a finite-dimensional representation of the robust LP.

$$O^n_r = \bigcap_{i=1}^{p_r} \text{half-space } i$$

**and strong duality** gives a finite-dimensional representation of the robust LP.
Finite-Dimensional Outer Approximation

The hyperplane representation of the outer polyhedral cones $O^n_r$

$$O^n_r = \bigcap_{j=1}^{p_r} \{ X \in S^n \mid u_j^\top X u_j \geq 0, \ u_j \in \Delta_r \}$$

and strong duality gives a finite-dimensional representation of the robust LP.

Theorem: The robust LP over $O^n_r$ admits an equivalent reformulation as a finite-dimensional conic linear program:

minimize $c^\top x$

subject to $x \in R^m$, $\mu_j \in R_+$, $\lambda_j \in K^*$, $\forall j = 1, \ldots, p_r$

$u_j^\top A_i(x)u_j = \mu_j + e_i^\top B^\top \lambda_j$, $\forall i = 1, \ldots, k$

$u_j^\top X u_j \geq 0$, $u_j \in \Delta_r$, $\forall j = 1, \ldots, p_r$

Its optimal value is a lower bound to the optimal value of the robust SDP.
A Challenge with the Inner Approximation

The **vertex representation** of the inner polyhedral cone $I^r_n = (O^r_n)^*$

$$I^r_n = \text{cone} \left\{ u_1 u_1^T, \ldots, u_p u_p^T \right\}$$

precludes a direct finite-dim. reformulation for the robust LP over $I^r_n$
A Challenge with the Inner Approximation

The **vertex representation** of the inner polyhedral cone \( I^n_r = (O^n_r)^* \)

\[
I^n_r = \text{cone}\{u_1 u_1^T, \ldots, u_{pr} u_{pr}^T\}
\]

**precludes a direct finite-dim.** reformulation for the robust LP over \( I^n_r \)

**Hyperplane representation of \( I^n_r \)**

- There exists \( q_r < \infty \) such that

\[
I^n_r = \bigcap_{i=1}^{q_r} \text{half-space}_i
\]
The **vertex representation** of the inner polyhedral cone $I^n_r = (O^n_r)^*$

\[ I^n_r = \text{cone}\{u_1 u_1^\top, \ldots, u_p u_p^\top\} \]

precludes a **direct finite-dim.** reformulation for the robust LP over $I^n_r$

**Hyperplane representation of $I^n_r$**

- There exists $q_r < \infty$ such that

\[ I^n_r = \bigcap_{i=1}^{q_r} \text{half-space } i \]

- Yields a finite-dim. reformulation of robust LP over $I^n_r$
Finite-Dimensional Outer Approximation

Let

\[ I^n_r = \bigcap_{j=1}^{q_r} \{ X \in S^n \mid \text{tr}(H_j X) \geq 0 \} \]

be the hyperplane representation of the inner polyhedral cone \( I^n_r \).
Finite-Dimensional Outer Approximation

Let

\[ \mathbf{I}_r^n = \bigcap_{j=1}^{q_r} \{ X \in \mathbb{S}^n \mid \text{tr}(H_i X) \geq 0 \} \]

be the hyperplane representation of the inner polyhedral cone \( \mathbf{I}_r^n \).

**Theorem:** The robust LP over \( \mathbf{I}_r^n \) admits an equivalent reformulation as a finite-dimensional conic linear program:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x \in \mathbb{R}^m, \mu_j \in \mathbb{R}_+, \lambda_j \in \mathbb{K}^*, \quad \forall \ j = 1, \ldots, q_r \\
\text{tr}(A_i(x)H_j) & = \mu_j + e_i^T B^\top \lambda_j, \quad \forall \ i = 1, \ldots, k \\
\end{align*}
\]

Its optimal value is an **upper bound** to the optimal value of the robust SDP.
A hyperplane representation of $\mathbb{I}_r^n$ can be

- **Computationally expensive** to compute
- **Impractical**: the number, $q_r$ of hyperplanes can be rather large
A hyperplane representation of $I^n_r$ can be

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**Question**: Can we work directly with the vertex representation of $I^n_r$?

$$I^n_r = \text{cone} \left\{ u_1u_1^\top, \ldots, u_p u_p^\top \right\}$$
A hyperplane representation of $I^n_r$ can be

- **Computationally expensive** to compute
- **Impractical**: the number, $q_r$ of hyperplanes can be rather large

**Question**: Can we work directly with the vertex representation of $I^n_r$?

$$I^n_r = \text{cone} \left\{ u_1 u_1^T, \ldots, u_{p_r} u_{p_r}^T \right\}$$

$$\exists \phi_1, \ldots, \phi_{p_r} : \mathbb{R}^k \to \mathbb{R}_+, \text{ such that }$$

$$\sum_{i=1}^k \xi_i A_i(x) = \sum_{j=1}^{p_r} \phi_j(\xi) u_j u_j^T, \quad \forall \xi \in \Xi.$$
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- **Computationally expensive** to compute
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**Question**: Can we work directly with the vertex representation of $\mathbb{I}_r^n$?

\[ \mathbb{I}_r^n = \text{cone}\left\{ u_1 u_1^T, \ldots, u_{p_r} u_{p_r}^T \right\} \]

\[ \sum_{i=1}^{k} \xi_i A_i(x) \quad \forall \, \xi \in \Xi \]

\[ \exists \, \phi_1, \ldots, \phi_{p_r} : \mathbb{R}^k \rightarrow \mathbb{R}_+, \text{ such that } \]

\[ \sum_{i=1}^{k} \xi_i A_i(x) = \sum_{j=1}^{p_r} \phi_j(\xi) u_j u_j^T, \quad \forall \, \xi \in \Xi. \]

Restriction to affine functions yields a finite-dim. inner approximation to robust LP
Inner Approximation to Robust LP over $I^n_r$

Let

$$I^n_r = \text{cone}\left\{u_1 u_1^T, \ldots, u_{p_r} u_{p_r}^T\right\}$$

be the vertex representation of the inner polyhedral cone $I^n_r$

**Theorem:** The robust LP over $I^n_r$ admits an finite-dimensional inner approximation as a conic linear program:

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x \in \mathbb{R}^m, \mu_j \in \mathbb{R}_+, \lambda_j \in K^*, \quad \forall \ j = 1, \ldots, p_r \\
& \quad A_i(x) = \sum_{j=1}^{p_r} e_i^T (\mu_j e_1 + B^T \lambda_j) u_j u_j^T, \quad \forall \ i = 1, \ldots, k
\end{align*}$$

Its optimal value is an **upper bound** to the optimal value of the robust SDP.
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Robust Resistance Network Design Problem

Given a circuit topology and a set $\mathcal{I} = \{Q\xi \mid \xi \in \Xi\}$ of external currents.
Robust Resistance Network Design Problem

Given a circuit topology and a set \( \mathcal{I} = \{ Q \xi \mid \xi \in \Xi \} \) of external currents

\[
\begin{align*}
\text{Objective: } & \text{Choose a conductance } g_{ij} \text{ for each line } (i, j) \text{ such that:} \\
\text{minimize } & \quad g \sum_{(i, j) \in \mathcal{E}} g_{ij} \\
\text{subject to } & \quad 1^\top g \leq b \quad \text{budget constraint} \\
& \quad g \geq 0 \quad \text{physical constraints}
\end{align*}
\]
Robust Resistance Network Design Problem

Given a circuit topology and a set $\mathcal{I} = \{Q\xi \mid \xi \in \Xi\}$ of external currents

Objective: Choose a conductance $g_{ij}$ for each line $(i, j)$ such that:

$$
\begin{align*}
\text{minimize} & \quad \tau \\
\text{subject to} & \quad \mathbf{1}^\top g \leq \omega \\
& \quad g \geq 0 \\
& \quad \begin{bmatrix}
\tau & Q\xi \\
Q\xi & M\text{diag}(g)M^\top
\end{bmatrix} \preceq 0, \quad \forall \xi \in \Xi
\end{align*}
$$
Unstructured Normed-Bounded Uncertainty

$$\Xi = \{ \xi \in \mathbb{R}^6 \mid \|\xi\|_2 \leq 2, \; \xi_1 = 1 \}$$
Unstructured Normed-Bounded Uncertainty

\[ \Xi = \{ \xi \in \mathbb{R}^6 \mid \|\xi\|_2 \leq 2, \; \xi_1 = 1 \} \]

### Robust LP Hierarchies

<table>
<thead>
<tr>
<th>Lower Bound</th>
<th>Upper Bound I</th>
<th>Upper Bound II</th>
</tr>
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<tbody>
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<td>( O^n_r )</td>
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Comparisons:
- **Ben-Tal et. al [‘00]** – (Optimal Value to Robust SDP)
  
  2.37
Unstructured Normed-Bounded Uncertainty

\[ \Xi = \{ \xi \in \mathbb{R}^6 \mid \| \xi \|_2 \leq 2, \xi_1 = 1 \} \]

### Robust LP Hierarchies

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### Comparisons:

- **Ben-Tal et. al [‘00]** – (Optimal Value to Robust SDP)
  
  \[ 2.37 \]
Unstructured Normed-Bounded Uncertainty

\[ \Xi = \{ \xi \in \mathbb{R}^6 \mid \|\xi\|_2 \leq 2, \ \xi_1 = 1 \} \]

### Robust LP Hierarchies

<table>
<thead>
<tr>
<th></th>
<th>Level ( r ) in Hierarchy</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Lower Bound ((O^n_r))</td>
<td>0</td>
</tr>
<tr>
<td>Upper Bound I ((I^n_r))</td>
<td>(\infty)</td>
</tr>
<tr>
<td>Upper Bound II ((I^n_r))</td>
<td>(\infty)</td>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>Level $r$ in Hierarchy</strong></td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Lower Bound</td>
<td>(0, 0)</td>
<td>(4.75, 3.15)</td>
<td>(6.72, 4.94)</td>
</tr>
<tr>
<td>Upper Bound I</td>
<td>$\infty$</td>
<td>comp. expensive</td>
<td></td>
</tr>
<tr>
<td>Upper Bound II</td>
<td>$\infty$</td>
<td></td>
<td></td>
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**Comparisons:**

- Ben-Tal et. al [‘00] – (Optimal Value to Robust SDP)
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\[ \Xi = \{ \xi \in \mathbb{R}^6 \mid \| (\xi_2, \xi_3) \|_2 \leq 1, \| (\xi_4, \xi_5, \xi_6) \|_2 \leq 1, \xi_1 = 1 \} \]
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<tr>
<th>Level in Hierarchy</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower Bound ((O^n_r))</td>
<td>0</td>
<td>1.65</td>
<td>3.66</td>
<td>4.19</td>
<td>4.24</td>
</tr>
<tr>
<td>Upper Bound I ((I^n_r))</td>
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<td>6.35</td>
<td>5.26</td>
<td>comp. expensive</td>
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<tr>
<td>Upper Bound II ((I^n_r))</td>
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<td>8.02</td>
<td>6.80</td>
<td>6.61</td>
<td>6.51</td>
</tr>
</tbody>
</table>

### Comparisons:
- **Ben-Tal et. al [’00]** – (Upper Bound to Robust SDP)
  - \(\infty\)
- **Scherer, Hol [’06]** – (Upper Bound to Robust SDP)
  - 4.27
Polytopic Uncertainty

\[ \Xi = \{ \xi \in \mathbb{R}^6 \mid \|\xi\|_{\infty} \leq 1, \ L\xi \geq 0, \ \xi_1 = 1 \} \]
Polytopic Uncertainty

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<tr>
<td>Level (r) in Hierarchy</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Lower Bound</td>
<td>0</td>
<td>3.40</td>
<td>8.17</td>
</tr>
<tr>
<td>Upper Bound I</td>
<td>(\infty)</td>
<td>8.96</td>
<td>8.44</td>
</tr>
<tr>
<td>Upper Bound II</td>
<td>(\infty)</td>
<td>8.96</td>
<td>8.44</td>
</tr>
</tbody>
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**Comparisons:**

- **Nemirovski, El-Ghaoui [‘00]** – (Not Applicable)
- **Scherer, Hol [‘06]** – (Upper Bound to Robust SDP)
  
  8.22
Summary and Future Research

- Developed computationally tractable **inner and outer hierarchies** to robust SDPs that are exact in the limit.

- **Approach**: Developed inner and outer polyhedral hierarchies to $S_+^n$.

- **Challenges**: Impractical for moderate levels in the hierarchy!
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**Future Research**

• Adaptively improve the polyhedral approx. of $S^n_+$ by using the guidance of the objective function!
Questions?

Thank you!
Raphael Louca
e-mail: rl553@cornell.edu